# The Effects of Changing Design Size, Axial Distances and Increased Center Points for Equiradial Design with Variation in Model Parameters

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***Abstract***
*It has been shown widely in literatures that axial distances, design sizes and increased center points affects designs in different settings and that when trying to minimize the trace of the information matrix the equiradial design for reduced model perform better than the full model correspondingly. This paper was presented to know the effect of changing axial distance, design size and increasing center point on the second-order equiradial design. The D-, G-, A-, E-, and T- optimality criteria were considered for full and reduced bivariate quadratic models alongside their efficiency criteria of D-, G-, A- and E-. The full model will be made reduced by omitting the interaction term and the two models shall be compared by the use of these criteria with center points from 1 to 5 inclusive. The results show that the relationship between D- and G- optimality criteria suggests that larger value of D-optimal design has smaller value of G-optimal design which in turn implies larger value of A-optimal and E-optimal designs. Also, A-optimality maintains a steady flow which is to say, constantly decreasing as center points increases, hence, we proposed A-optimality criterion as the best criterion among the once studied for a reduced quadratic model. The D-Optimality of equiradial designs increases for increasing axial distances for radial points n=5 and 1center point for a reduced model which is also true for full model. It was also observed that D-Optimality of equiradial design for axial distance of 1.414 shows superiority over equiradial design of 1.0 both for full and reduced model and it is true for all the radial and center points studied. The D-Optimality of equiradial design is better for reduced model than for full model for all axial distance and center points studied. This implies that equiradial design minimizes the variance of parameter estimates for reduced model than for full model.*

***Keywords;*** *equiradial designs; second-order models; full and reduced quadratic model; optimality criteria; efficiency criteria.*

## 1. Introduction

Some designs in Response Surface Methodology such as First and Second-order designs play vital roles in modelling response functions. Factorial designs are the most classical designs and they assume the regression models used as an approximating model to the true unknown response function, which is known as a full polynomial model. Though, models with improper polynomial regression functions also exist. Due to the plasticity of optimal design theory, it is possible to get properties of designs for polynomial regression functions having complete terms present or some terms missing. Iwundu & Albert-Udochukwuka (2014) considered the behavior of D-optimal exact designs for first-order polynomial models under changing regression polynomials with or without intercept terms or with or without interactive terms. The importance of first order models and designs cannot be much talked about, as they can be applied severally in the industrial processes, mostly in screening experiments. Similarly, the second-order response surface methodology designs are mostly very essential in modelling second-order response functions in the presence of curvature. This designs are the central composite designs, $3^k$ factorial designs, the Box -Behnken designs and even D-optimal designs which have been studied extensively. Nonetheless, equiradial designs do exist and some can serve practically well in modelling second-order effects. This is as seen in Iwundu and Onu (2017).

Iwundu (2016b) considered on the behavior of second-order N-point equiradial designs with varying model parameters, the study looked at equiradial design for axial distance of 1.0 for quadratic models one with full parameters represented and the other with interaction effect omitted. The behavior of this design was studied for this two models for one center point. Also, Iwundu and Onu (2017) studied equiradial designs for axial distances of 1.0 and 1.414 and compared them with the central composite designs inscribed and face centered using quadratic model with all the parameters represented. This papers did not look at equiradial designs for axial distance of 1.414 with reduced quadratic model, i.e quadratic model having the interaction term omitted and the behavior of these designs with reduced model for increasing center points from 1 to 5 inclusive. Hence this paper, to study the effect of changes in design size, axial distances and increased center points with the variations in model parameters.

The work is aimed at investigating the effect of changing design size, axial distance and increased center points with variation in model parameters.

## 2. Literature Review

Myer et al. (2009) describes equiradial designs as interesting and special design with two-factors for use in modelling second-order response functions which have design points seem on a common sphere. Equiradial designs require not many experimental runs and is initiated with a pentagon of equally spaced points on the sphere. The matrix formed from the design is expressed as seen in Iwundu (2016a and b), Iwundu and Onu (2017) and Myer et al (2009) as

$$x_1 = \{\rho Cos\left(\theta + \frac{2\Pi u}{n_1}\right), x_2 = \{\rho Sin\left(\theta + \frac{2\Pi u}{n_1}\right)\} \quad (1)$$

Where $u = 0, 1, 2, 3, \ . \ . \ . \ , n_1 - 1$

Also $x_1$ and $x_2$ represent the two regulatory variables, ρ is the radius of the design and $n_1$ represents the number of points on the sphere. $n_c$ center points are usually added to the $n_1$radial points of the design. Myer et al. (2009) saw that the value of $\theta$ could be assumed equal to zero because the information matrix $X'X$, of the design is unaffected by the value of $\theta$. Some eminent researchers have considered second-order response surface models and designs.

Onukogu and Iwundu (2007), studied the construction of efficient and optimal experimental designs for second-order response surface models.

Iwundu and Onu (2017) examined the preferences of equiradial designs under changing design size, axial distances and increased center points and their relationships to the N-point central composite designs. While some others have considered optimality of designs for second-order models as seen Iwundu and Onu (2017) and Dette and Grigoriev (2014)).

Chigbu and Nduka (2006), Chigbu et al. (2009), Ukaegbu and Chigbu (2015), Oyejola and Nwanya (2015) and Iwundu (2015) have in different occasions considered optimal choices of design points for second-order response surface designs and performances of several types of second-order response surface designs. Graphical techniques have been used in studying the variance properties of second-order response surface designs as seen in Myer et al. (1992), Giovannitti-Jensen and Myers (1989), Zahran et al. (2003). In a recent study by Iwundu (2016a), the equiradial designs were seen comparable with the standard central composite designs. Particularly, the D-efficiency values reveal that the N-point spherical equiradial designs are better than the inscribed central composite design though inferior to the circumscribed central composite design with efficiency values being less than 50% in all cases studied. This work considers the behaviour of the equiradial designs for changing the parameters of the model with design radius or axial distance ρ = 1.0 and 1.414 for spherical region and increasing center points from 1 to 5. it also considers investigating the assertion that designs optimal for one model need not be optimal for another model

## 3. Materials and Methods

It is a common practice to assume some high order interaction terms of a model to be negligible. This practice is seen, for instance, in the area of factorial experiments where the highest order interaction is assumed negligible. It is even possible to assume a no-intercept (or zero-intercept) model as seen in origin regression. In studying the behaviour of alternative second-order N-point equiradial designs under variations of model parameters, a case where the second-order full model is assumed as well as a case where the highest order interaction is assumed negligible and is thus removed from the model shall be considered. The second-order full quadratic model (with intercept, main effects and interaction effects) to be employed in this study is

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \left\{\sum_{i=1}^{k} \sum_{j>1}^{k} \beta_{ij} \left(x_i x_j\right)\right\} + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \boldsymbol{\varepsilon} \quad (2)$$

$$; -p < x_i, x_j < p$$

Which is simplified as

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \boldsymbol{\varepsilon} \quad (3)$$

and the second -order reduced (without-interaction) model to be used in this research is the model

$$y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \boldsymbol{\varepsilon} \quad (4)$$

Which also is simplified as

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{11} x_1^2 + \beta_{22} x_2^2 + \boldsymbol{\varepsilon} \quad (5)$$

With intercept, main effects and quadratic effects only. Because equiradial designs are alternative second-order N-point spherical response surface methodology designs in two variables, k shall be set at 2 as seen in Iwundu and Onu (2017) and Iwundu (2016a). The real value ρ represents the design radius or the axial distance. Interest here lies in knowing what effect the removal of the interactive term would have on the N-point designs with change in axial distance and increased center point say $1 \leq c \leq 5$ inclusive for design sizes of 5, 6, 7 and 8 as measured by some optimality criteria and efficiency values. The four alphabetic optimality criteria to be employed in this work are A-, D-, E-and G-optimality criteria. Each shall summarize how good the experimental designs are, using the two defined models for each center point added for the two axial distances considered. The axial distance known as the design radius used are ρ=1.0 and 1.414 and these are chosen to remain in spherical region. As seen in some standard literatures on optimal designs such as Myer et al. (2009) and Rady et al. (2009), D-optimality criterion focuses on good model parameter estimation. A D-optimal design is one in which the determinant of the information matrix

$$\left|\frac{X'X}{N}\right|$$

is maximized over all designs, where $X$ represents the design matrix associated with the design and $X'$ represents the transpose of $X$. A-optimality criterion is a criterion that minimizes the sum of the variances of the model coefficients. It therefore minimizes the trace of $\left(\frac{X'X}{N}\right)^{-1}$ which is defined as

Min tr $\left(\frac{X'X}{N}\right)^{-1}$ where Min indicates that the minimization is over all designs and tr means trace.

The D-optimality criterion is equivalent to minimizing the determinant of $\left(\frac{X'X}{N}\right)^{-1}$.
The E-optimality criterion maximizes the minimum Eigen value of $\frac{X'X}{N}$ which is equivalent to minimizing the maximum Eigen value of $M^{-1}$ and it is given as

$$Max\ \lambda_{Min}\left(\frac{X'X}{N}\right) \equiv Min\ \lambda_{Max}\left(\left(\frac{X'X}{N}\right)^{-1}\right)$$

Where $\lambda_{Min}$ and $\lambda_{Max}$ represents minimum and maximum Eigen values of the design matrix respectively and $\frac{X'X}{N}$ which can be represented by $M$, so, we say that

$M = \frac{X'X}{N}$ and $\left(\frac{X'X}{N}\right)^{-1}$ is the inverse of the normalized information matrix given as $(M)^{-1}$

The G-optimality criterion is a criterion that minimizes the maximum scaled prediction variance, v$(x)$, in the design region and is given as

$$Min\left\{max_{\underline{x}\in R}V(\underline{x})\right\}$$

Iwundu (2016b) investigated the assertion that designs optimal for one model need not be optimal for another model using equiradial design with axial distance of 1.0 and one center point for full and reduced bivariate quadratic models, this study shall then investigate that assertion using equiradial designs with axial distances of 1.0 and 1.414 and center points addition from 1 to 5 inclusive, for the full and reduced bivariate models in equations 2 and 4. Furthermore, the efficiency of designs shall be considered. In comparing two designs, the relative efficiency is seen as the ratio of their separate efficiencies. The D-efficiency criterion shall be used as the test criterion and is given as

$$D_{eff} = \left(\det(\frac{X'X}{N})\right)^{\frac{1}{P}} \tag{6}$$

While the relative efficiency of design1 in relation to design2 is given as

$$D_{rel,eff} = \left(\frac{det\left(\frac{X'X}{N}\right)_1}{det\left(\frac{X'X}{N}\right)_2}\right)^{\frac{1}{p}} \tag{7}$$

Note that $det\left(\frac{X'X}{N}\right)_1$ is the determinant of normalized information matrix of design1and $\left(\frac{X'X}{N}\right)_2$ is the determinant of normalized information matrix of design 2. We normalize an information matrix in order to over the effect of change in design sizes.

## 4. Results and Discussion

From equation (1) we obtain the measures for equradial designs for each of radius 1.0 and 1.414 for each of design sizes 5, 6, 7 and 8 for each center point addition, as seen in Iwundu and Onu (2017) and Iwundu (2016b).

Equiradial designs uses two explanatory variables $x_1$and $x_2$, hence applying the models in (3) and (5) known as full and reduced models respectively, using Math Lab the results are as seen

Equiradial design for n=5 and ρ=1.0 with a reduced quadratic model

The matrix for equiradial design for radial points n=5 with radius 1.0 and 1 center point is given as

$$X = \begin{pmatrix} 1.0000 & 1.0000 & 0 & 1.0000 & 0 \\ 1.0000 & 0.3090 & 0.9510 & 0.0955 & 0.9044 \\ 1.0000 & -0.8100 & 0.5870 & 0.6561 & 0.3446 \\ 1.0000 & -0.8080 & -0.5890 & 0.6529 & 0.3469 \\ 1.0000 & 0.3110 & -0.9500 & 0.0967 & 0.9025 \\ 1.0000 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The transpose of $X$ is given as

$$X' = \begin{pmatrix} 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ 1.0000 & 0.3090 & -0.8100 & -0.8080 & 0.3110 & 0 \\ 0 & 0.9510 & 0.5870 & -0.5890 & -0.9500 & 0 \\ 1.0000 & 0.0955 & 0.6561 & 0.6529 & 0.0967 & 0 \\ 0 & 0.9044 & 0.3446 & 0.3469 & 0.9025 & 0 \end{pmatrix}$$

The information matrix is obtained as

$$X'X = \begin{pmatrix} 6.0000 & 0.0020 & -0.0010 & 2.5012 & 2.4984 \\ 0.0020 & 2.5012 & -0.0011 & 0.0006 & 0.0007 \\ -0.0010 & -0.0011 & 2.4984 & -0.0005 & 0.0007 \\ 2.5012 & 0.0006 & -0.0005 & 1.8752 & 0.6262 \\ 2.4984 & 0.0007 & 0.0007 & 0.6262 & 1.8715 \end{pmatrix}$$

The determinant of $X'X$ is given as

$|X'X| = 19.4801$

The trace of $X'X$ is obtained as seen

tr $(X'X)$= 14.7463

we get the Eigen value of $X'X$ as

Eigen $(X'X)$=

$$\begin{pmatrix} 0.3050 \\ 1.2471 \\ 2.4980 \\ 2.5016 \\ 8.1946 \end{pmatrix}$$

The inverse of $X'X$ is obtained as

$$(X'X)^{-1} = \begin{pmatrix} 1.0000 & -0.0003 & 0.0005 & -0.9997 & -1.0004 \\ -0.0003 & 0.3998 & 0.0002 & 0.0002 & 0.0001 \\ 0.0005 & 0.0002 & 0.4003 & -0.0003 & -0.0007 \\ -0.9997 & 0.0002 & -0.0003 & 1.5998 & 0.7993 \\ -1.0004 & 0.0001 & -0.0007 & 0.7993 & 1.6024 \end{pmatrix}$$

The trace of $(X'X)^{-1}$ is obtained as
tr$((X'X)^{-1}) = 5.002$

We normalize the information matrix as seen

$$M = \frac{X'X}{N} = \begin{pmatrix} 1.0000 & 0.0003 & -0.0002 & 0.4169 & 0.4164 \\ 0.0003 & 0.4169 & -0.0002 & 0.0001 & 0.0001 \\ -0.0002 & -0.0002 & 0.4164 & -0.0001 & 0.0001 \\ 0.4169 & 0.0001 & -0.0001 & 0.3125 & 0.1044 \\ 0.4164 & 0.0001 & 0.0001 & 0.1044 & 0.3119 \end{pmatrix}$$

The determinant of the normalized information matrix is obtained as
$|M| = 0.0025$
We obtain the trace of $M$ given as
tr $(M)$=2.4577
the eigen values of $M$ is given as

$$\text{eigen } (M) = \begin{pmatrix} 0.0508 \\ 0.2079 \\ 0.4163 \\ 0.4169 \\ 1.3658 \end{pmatrix}$$

The inverse of $M$ is

$$M^{-1} = \begin{pmatrix} 6.0000 & -0.0016 & 0.0029 & -5.9984 & -6.0026 \\ -0.0016 & 2.3989 & 0.0011 & 0.0011 & 0.0009 \\ 0.0029 & 0.0011 & 2.4015 & -0.0019 & -0.0041 \\ -5.9984 & 0.0011 & -0.0019 & 9.5988 & 4.7957 \\ -6.0026 & 0.0009 & -0.0041 & 4.7957 & 9.6144 \end{pmatrix}$$

The scaled prediction variance of $M^{-1}$is
V=[6 5.1761 3.8358 3.8249 5.1632 6]
Equiradial design for n=5 and ρ=1.414 and 1 center (c) with a full quadratic model
The matrix for equiradial design for n=5 and ρ=1.414 and 1 center point (c) with a full quadratic model is given as

$$X = \begin{pmatrix} 1.0000 & 1.4140 & 0 & 0 & 1.9990 & 0 \\ 1.0000 & 0.4360 & 1.3450 & 0.5860 & 0.1900 & 1.8090 \\ 1.0000 & -1.1450 & 0.8300 & -0.9500 & 1.3110 & 0.6890 \\ 1.0000 & -1.1430 & -0.8300 & 0.9520 & 1.3060 & 0.6940 \\ 1.0000 & 0.4400 & -1.3440 & -0.5910 & 0.1940 & 1.8060 \\ 1.0000 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The transpose of $X$ is given as

$$X' = \begin{pmatrix} 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ 1.4140 & 0.4360 & -1.1450 & -1.1430 & 0.4400 & 0 \\ 0 & 1.3450 & 0.8300 & -0.8300 & -1.3440 & 0 \\ 0 & 0.5860 & -0.9500 & 0.9520 & -0.5910 & 0 \\ 1.9990 & 0.1900 & 1.3110 & 1.3060 & 0.1940 & 0 \\ 0 & 1.8090 & 0.6890 & 0.6940 & 1.8060 & 0 \end{pmatrix}$$

We obtain the information matrix as seen

$$X'X = \begin{pmatrix} 6.0000 & 0.0020 & 0.0010 & -0.0030 & 5.0000 & 4.9980 \\ 0.0020 & 5.0006 & -0.0066 & -0.0049 & 0.0009 & 0.0012 \\ 0.0010 & -0.0066 & 4.9932 & 0.0038 & -0.0010 & 0.0017 \\ -0.0030 & -0.0049 & 0.0038 & 2.5015 & -0.0055 & -0.0011 \\ 5.0000 & 0.0009 & -0.0010 & -0.0055 & 7.4941 & 2.5037 \\ 4.9980 & 0.0012 & 0.0017 & -0.0011 & 2.5037 & 7.4905 \end{pmatrix}$$

The determinant of $X'X$ is obtained as

$|X'X|$ =3.1145e+003

We obtain the trace of $X'X$ as

tr $(X'X)$= 33.4798

the eigen values of $X'X$ is

$$\text{eigen } (X'X) = \begin{pmatrix} 0.6514 \\ 2.5015 \\ 4.9871 \\ 4.9907 \\ 5.0045 \\ 15.3446 \end{pmatrix}$$

The inverse of $X'X$ is given as

$$(X'X)^{-1} = \begin{pmatrix} 1.0000 & -0.0002 & -0.0001 & -0.0001 & -0.5001 & -0.5001 \\ -0.0002 & 0.2000 & 0.0003 & 0.0004 & 0.0001 & 0.0001 \\ -0.0001 & 0.0003 & 0.2003 & -0.0003 & 0.0001 & 0.0000 \\ -0.0001 & 0.0004 & -0.0003 & 0.3998 & 0.0004 & 0.0000 \\ -0.5001 & 0.0001 & 0.0001 & 0.0004 & 0.4003 & 0.1999 \\ -0.5001 & 0.0001 & 0.0000 & 0.0000 & 0.1999 & 0.4004 \end{pmatrix}$$

The trace of $(X'X)^{-1}$ is obtained as
$\text{tr}((X'X)^{-1}) = 2.601$

Normalizing the information matrix, we have

$$M = \begin{pmatrix} 1.0000 & 0.0003 & 0.0002 & -0.0005 & 0.8333 & 0.8330 \\ 0.0003 & 0.8334 & -0.0011 & -0.0008 & 0.0002 & 0.0002 \\ 0.0002 & -0.0011 & 0.8322 & 0.0006 & -0.0002 & 0.0003 \\ -0.0005 & -0.0008 & 0.0006 & 0.4169 & -0.0009 & -0.0002 \\ 0.8333 & 0.0002 & -0.0002 & -0.0009 & 1.2490 & 0.4173 \\ 0.8330 & 0.0002 & 0.0003 & -0.0002 & 0.4173 & 1.2484 \end{pmatrix}$$

The determinant of $M$ is obtained as
$|M|= 0.0668$
The trace of $M$ is given as
$\text{tr}\,(M)=5.5800$
Eigen values of $M$ is

$$\text{Eigen}\,(M)= \begin{pmatrix} 0.1086 \\ 0.4169 \\ 0.8312 \\ 0.8318 \\ 0.8341 \\ 2.5574 \end{pmatrix}$$

The inverse of $M$ is given as

$$M^{-1}= \begin{pmatrix} 6.0000 & -0.0011 & -0.0008 & -0.0007 & -3.0007 & -3.0005 \\ -0.0011 & 1.1999 & 0.0016 & 0.0024 & 0.0005 & 0.0004 \\ -0.0008 & 0.0016 & 1.2016 & -0.0018 & 0.0007 & 0.0000 \\ -0.0007 & 0.0024 & -0.0018 & 2.3986 & 0.0022 & 0.0001 \\ -3.0007 & 0.0005 & 0.0007 & 0.0022 & 2.4020 & 1.1993 \\ -3.0005 & 0.0004 & 0.0000 & 0.0001 & 1.1993 & 2.4022 \end{pmatrix}$$

The scaled prediction variances are obtained as
$V = (6 \quad 6 \quad 6 \quad 6 \quad 6 \quad 6)$
These processes continue for n=6, 7 and 8 for both equiradial designs of 1.0 and 1.414, with increasing center points in each case. See the summary of the results in table 1and 2.

Table 1: Comparison of Optimal values for Equiradial designs for ρ=1.0 and 1.414 for Reduced Model

| Equiradial design for ρ=1.0 | | | | | | | | | | Equiradial design for ρ=1.414 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Reduced Model** | | | | | | | | | | **Reduced Model** | | | | | | | | | |
| n | C | N | Det | tr(M) | tr($M^{-1}$) | Max V | Min V | Max Eig | Min Eig | n | C | N | Det | tr(M) | tr($M^{-1}$) | Max V | Min V | Max Eig | Min Eig |
| 5 | 1 | 6 | 0.0025 | 2.457 | 5.002 | 6 | 3.825 | 1.366 | 0.051 | 5 | 1 | 6 | 0.160 | 5.163 | 2.201 | 6 | 3.824 | 2.557 | 0.109 |
| | 2 | 7 | 0.0023 | 2.249 | 3.502 | 7 | 3.5 | 1.277 | 0.080 | | 2 | 7 | 0.148 | 4.568 | 1.451 | 7 | 3.5 | 2.246 | 0.182 |
| | 3 | 8 | 0.0018 | 2.093 | 3.002 | 8 | 2.667 | 1.216 | 0.096 | | 3 | 8 | 0.114 | 4.122 | 1.201 | 8 | 2.667 | 2.017 | 0.232 |
| | 4 | 9 | 0.0013 | 1.972 | 2.752 | 9 | 2.25 | 1.173 | 0.105 | | 4 | 9 | 0.084 | 3.775 | 1.076 | 9 | 2.25 | 1.843 | 0.267 |
| | 5 | 10 | $9.74 \times 10^{-4}$ | 1.875 | 2.602 | 10 | 2 | 1.140 | 0.110 | | 5 | 10 | 0.062 | 3.498 | 1.001 | 10 | 2 | 1.707 | 0.293 |
| 6 | 1 | 7 | 0.0024 | 2.410 | 4.665 | 7 | 4.074 | 1.384 | 0.044 | 6 | 1 | 7 | 0.154 | 5.283 | 2.084 | 7 | 4.0743 | 2.620 | 0.093 |
| | 2 | 8 | 0.0025 | 2.312 | 3.167 | 6.665 | 4 | 1.303 | 0.072 | | 2 | 8 | 0.158 | 4.748 | 1.334 | 6.670 | 4 | 2.339 | 0.160 |
| | 3 | 9 | 0.0021 | 2.167 | 2.667 | 7.498 | 3 | 1.244 | 0.089 | | 3 | 9 | 0.131 | 4.331 | 1.084 | 7.493 | 3 | 2.123 | 0.209 |
| | 4 | 10 | 0.0016 | 2.050 | 2.417 | 8.335 | 2.5 | 1.200 | 0.100 | | 4 | 10 | 0.104 | 3.998 | 0.959 | 8.337 | 2.5 | 1.954 | 0.246 |
| | 5 | 11 | 0.0013 | 1.955 | 2.267 | 9 | 2.2 | 1.166 | 0.136 | | 5 | 11 | 0.080 | 3.726 | 0.884 | 9.161 | 2.2 | 1.818 | 0.273 |
| 7 | 1 | 8 | 0.0023 | 2.562 | 4.090 | 7.452 | 3.575 | 1.458 | 0.046 | 7 | 1 | 8 | 0.148 | 5.372 | 2.004 | 7.995 | 3.535 | 2.645 | 0.082 |
| | 2 | 9 | 0.0024 | 2.388 | 2.858 | 6.967 | 3.993 | 1.373 | 0.074 | | 2 | 9 | 0.164 | 4.886 | 1.251 | 6.361 | 3.976 | 2.393 | 0.143 |
| | 3 | 10 | 0.0021 | 2.249 | 2.427 | 7.737 | 3.254 | 1.309 | 0.093 | | 3 | 10 | 0.145 | 4.497 | 1.00 | 7.068 | 3.333 | 2.193 | 0.189 |
| | 4 | 11 | 0.0018 | 2.136 | 2.208 | 8.508 | 2.700 | 1.259 | 0.105 | | 4 | 11 | 0.120 | 4.179 | 0.875 | 7.775 | 2.750 | 2.031 | 0.225 |
| | 5 | 12 | 0.0014 | 2.041 | 2.075 | 9.280 | 2.365 | 1.219 | 0.110 | | 5 | 12 | 0.097 | 3.915 | 0.799 | 8.481 | 2.400 | 1.899 | 0.253 |

Table 2: Comparison of Optimal values for Equiradial designs for ρ=1.0 and 1.414 for Full Model

| Equiradial design for ρ=1.0 | | | | | | | | | | Equiradial design for ρ=1.414 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Full Model** | | | | | | | | | | **Full Model** | | | | | | | | | |
| n | c | N | Det | tr(M) | tr($M^{-1}$) | Max V | Min V | Max Eig | Min Eig | n | C | N | Det | tr(M) | tr($M^{-1}$) | Max V | Min V | Max Eig | Min Eig |
| 5 | 1 | 6 | $2.61 \times 10^{-4}$ | 2.562 | 6.600 | 6 | 6 | 1.366 | 0.051 | 5 | 1 | 6 | 0.0668 | 5.580 | 2.601 | 6 | 6 | 2.557 | 0.109 |
| | 2 | 7 | $2.07 \times 10^{-4}$ | 2.339 | 5.100 | 7 | 3.5 | 1.277 | 0.080 | | 2 | 7 | 0.0529 | 4.926 | 1.851 | 7 | 3.5 | 2.246 | 0.182 |
| | 3 | 8 | $1.395 \times 10^{-4}$ | 2.172 | 4.600 | 8 | 2.667 | 1.216 | 0.078 | | 3 | 8 | 0.0356 | 4.435 | 1.601 | 8 | 2.6667 | 2.017 | 0.232 |
| | 4 | 9 | $9.175 \times 10^{-5}$ | 2.041 | 4.350 | 9 | 2.25 | 1.173 | 0.070 | | 4 | 9 | 0.0234 | 4.053 | 1.476 | 9 | 2.25 | 1.843 | 0.268 |
| | 5 | 10 | $6.095 \times 10^{-5}$ | 1.937 | 4.200 | 10 | 2 | 1.140 | 0.063 | | 5 | 10 | 0.0156 | 3.748 | 1.401 | 10 | 2 | 1.707 | 0.250 |
| 6 | 1 | 7 | $2.584 \times 10^{-4}$ | 2.607 | 5.999 | 7 | 5.831 | 1.384 | 0.044 | 6 | 1 | 7 | 0.0660 | 5.712 | 2.417 | 7 | 5.829 | 2.620 | 0.093 |
| | 2 | 8 | $2.320 \times 10^{-4}$ | 2.406 | 4.499 | 6.671 | 4 | 1.303 | 0.072 | | 2 | 8 | 0.0592 | 5.123 | 1.667 | 6.67 | 4 | 2.339 | 0.160 |
| | 3 | 9 | $1.717 \times 10^{-4}$ | 2.250 | 3.999 | 7.505 | 3 | 1.244 | 0.083 | | 3 | 9 | 0.0438 | 4.665 | 1.417 | 7.504 | 3 | 2.123 | 0.209 |
| | 4 | 10 | $1.216 \times 10^{-4}$ | 2.125 | 3.749 | 8.339 | 2.5 | 1.200 | 0.075 | | 4 | 10 | 0.0310 | 4.298 | 1.292 | 8.338 | 2.5 | 1.954 | 0.246 |
| | 5 | 11 | $8.582 \times 10^{-5}$ | 2.023 | 3.599 | 9.173 | 2.2 | 1.167 | 0.068 | | 5 | 11 | 0.0219 | 3.999 | 1.217 | 9.171 | 2.2 | 1.818 | 0.273 |

| | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 7 | 1 | 8 | $2.502 \times 10^{-4}$ | 2.640 | 5.572 | 8 | 5.710 | 1.398 | 0.039 | 7 | 1 | 8 | 0.0647 | 5.809 | 2.290 | 8 | 5.66 | 2.645 | 0.082 |
| | 2 | 9 | $2.468 \times 10^{-4}$ | 2.458 | 4.072 | 6.492 | 4.5 | 1.324 | 0.065 | | 2 | 9 | 0.0638 | 5.275 | 1.537 | 6.498 | 4.50 | 2.393 | 0.143 |
| | 3 | 10 | $1.967 \times 10^{-4}$ | 2.312 | 3.572 | 7.146 | 3.333 | 1.267 | 0.083 | | 3 | 10 | 0.0508 | 4.847 | 1.286 | 7.174 | 3.33 | 2.193 | 0.189 |
| | 4 | 11 | $1.481 \times 10^{-4}$ | 2.193 | 3.322 | 7.861 | 2.750 | 1.224 | 0.080 | | 4 | 11 | 0.0383 | 4.497 | 1.161 | 7.941 | 2.75 | 2.031 | 0.225 |
| | 5 | 12 | $1.098 \times 10^{-4}$ | 2.094 | 3.172 | 8.576 | 2.4 | 1.190 | 0.073 | | 5 | 12 | 0.0284 | 4.206 | 1.085 | 8.608 | 2.4 | 1.899 | 0.253 |
| 8 | 1 | 9 | $2.419 \times 10^{-4}$ | 2.668 | 5.246 | 9 | 5.616 | 1.410 | 0.035 | 8 | 1 | 9 | 0.0615 | 5.887 | 2.188 | 9 | 5.621 | 2.704 | 0.073 |
| | 2 | 10 | $2.571 \times 10^{-4}$ | 2.501 | 3.746 | 6..259 | 5 | 1.341 | 0.060 | | 2 | 10 | 0.0654 | 5.398 | 1.438 | 6.257 | 5 | 2.470 | 0.130 |
| | 3 | 11 | $2.177 \times 10^{-4}$ | 2.364 | 3.246 | 6.886 | 3.667 | 1.287 | 0.077 | | 3 | 11 | 0.0554 | 4.998 | 1.188 | 6.883 | 4 | 2.280 | 0.174 |
| | 4 | 12 | $1.722 \times 10^{-4}$ | 2.251 | 2.996 | 7.510 | 3 | 1.244 | 0.084 | | 4 | 12 | 0.0438 | 4.665 | 1.063 | 7.508 | 3 | 2.124 | 0.209 |
| | 5 | 13 | $1.332 \times 10^{-4}$ | 2.155 | 2.851 | 8.136 | 2.6 | 1.210 | 0.077 | | 5 | 13 | 0.0339 | 4.383 | 0.988 | 8.134 | 2.6 | 1.993 | 0.237 |

Table 3: Summary of the Computations of alphabetic efficiencies for equiradial Designs (ρ =1.0 and 1.414, $c \geq 1$) for reduced model

| n | N | Equiradial ρ=1.0 | | | | equiradial $\rho$=1.414 | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $Deff$ | $Geff$ | $Aeff$ | $Eeff$ | $Deff$ | $Geff$ | $Aeff$ | $Eeff$ |
| 5 | 1 | 30.17 | 83.33 | 100 | 100 | 69.31 | 83.33 | 100 | 100 |
| | 2 | 29.67 | 71.43 | 70.01 | 93.51 | 68.24 | 71.43 | 65.92 | 87.84 |
| | 3 | 28.25 | 62.50 | 60.01 | 89.04 | 64.77 | 62.50 | 54.57 | 78.88 |
| | 4 | 26.47 | 55.56 | 55.02 | 85.85 | 60.93 | 55.56 | 48.89 | 72.06 |
| | 5 | 24.99 | 50.00 | 52.02 | 83.50 | 57.34 | 50.00 | 45.48 | 66.74 |
| 6 | 1 | 29.93 | 71.43 | 100 | 100 | 68.79 | 71.43 | 100 | 100 |
| | 2 | 30.17 | 75.02 | 67.89 | 94.13 | 69.14 | 74.97 | 64.01 | 89.28 |
| | 3 | 29.14 | 66.68 | 57.17 | 89.86 | 66.60 | 66.73 | 52.02 | 81.05 |
| | 4 | 27.59 | 59.99 | 51.81 | 86.69 | 63.59 | 59.97 | 46.02 | 74.58 |
| | 5 | 26.47 | 55.56 | 48.60 | 84.26 | 60.34 | 54.58 | 42.42 | 69.38 |
| 7 | 1 | 29.67 | 67.09 | 100 | 100 | 68.24 | 62.54 | 100 | 100 |
| | 2 | 29.93 | 71.77 | 69.88 | 94.17 | 69.66 | 78.60 | 62.43 | 90.44 |
| | 3 | 29.14 | 67.85 | 59.34 | 89.75 | 67.96 | 70.74 | 49.90 | 82.88 |
| | 4 | 28.25 | 58.77 | 53.99 | 86.33 | 65.44 | 64.31 | 43.66 | 76.78 |
| | 5 | 26.87 | 53.88 | 50.73 | 83.64 | 62.71 | 58.95 | 39.87 | 71.77 |
| 8 | 1 | 29.41 | 55.56 | 100 | 100 | 67.39 | 55.56 | 100 | 100 |
| | 2 | 30.41 | 79.95 | 64.70 | 95.10 | 69.66 | 80.00 | 61.30 | 91.34 |
| | 3 | 29.93 | 72.68 | 52.93 | 91.28 | 68.61 | 72.70 | 48.40 | 84.32 |
| | 4 | 29.14 | 66.62 | 47.05 | 88.26 | 66.70 | 66.68 | 41.95 | 78.53 |
| | 5 | 27.93 | 61.50 | 43.52 | 85.84 | 64.31 | 61.50 | 38.08 | 73.70 |

Table 4: Summary of the Computations of alphabetic efficiencies for equiradial Designs ($\rho$ = 1.0 and 1.414, $c \geq 1$) for full model

| n | N | Equiradial $\rho$=1.0 | | | | Equiradial $\rho$=1.414 | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | $Deff$ | $Geff$ | $Aeff$ | $Eeff$ | $Deff$ | $Geff$ | $Aeff$ | $Eeff$ |
| 5 | 1 | 30.17 | 83.33 | 100 | 100 | 63.64 | 100 | 100 | 100 |
| | 2 | 29.67 | 71.43 | 77.27 | 93.51 | 61.21 | 85.71 | 71.16 | 87.84 |
| | 3 | 28.25 | 62.50 | 69.70 | 89.04 | 57.29 | 75 | 61.55 | 78.84 |
| | 4 | 26.47 | 55.56 | 65.91 | 85.85 | 53.41 | 66.67 | 56.75 | 72.08 |
| | 5 | 24.99 | 50.00 | 63.64 | 83.50 | 49.92 | 60 | 53.86 | 66.76 |
| 6 | 1 | 29.93 | 71.43 | 100 | 100 | 63.51 | 85.71 | 100 | 100 |
| | 2 | 30.17 | 75.02 | 75.00 | 94.13 | 62.37 | 89.96 | 68.97 | 89.27 |
| | 3 | 29.14 | 66.68 | 66.66 | 89.86 | 59.31 | 79.96 | 58.63 | 81.03 |
| | 4 | 27.59 | 59.99 | 62.49 | 86.69 | 55.98 | 71.96 | 53.45 | 74.58 |
| | 5 | 26.47 | 55.56 | 59.99 | 84.26 | 52.83 | 65.42 | 50.35 | 69.39 |
| 7 | 1 | 29.67 | 67.09 | 100 | 100 | 63.30 | 75 | 100 | 100 |
| | 2 | 29.93 | 71.77 | 73.08 | 94.17 | 63.15 | 92 | 67.12 | 90.47 |
| | 3 | 29.14<br>89.75 | 67.85 | 64.11 | | 60.80 | 83.64 | 56.16 | 82.91 |
| | 4 | 28.25 | 58.77 | 59.62 | 86.33 | 58.00 | 75.56 | 50.70 | 76.79 |
| | 5 | 26.87<br>83.64 | 53.88 | 56.93 | | 55.17 | 69.70 | 47.38 | 71.80 |
| 8 | 1 | 29.41 | 55.56 | 100 | 100 | 62.77 | 66.67 | 100 | 100 |
| | 2 | 30.41<br>95.10 | 79.95 | 71.41 | | 63.42 | 95.89 | 65.72 | 91.35 |
| | 3 | 29.93<br>91.28 | 72.68 | 61.88 | | 61.68 | 87.17 | 54.30 | 84.32 |
| | 4 | 29.14<br>88.26 | 66.62 | 57.11 | | 59.31 | 79.91 | 48.58 | 78.55 |
| | 5 | 27.93<br>85.84 | 61.50 | 54.35 | | 56.83 | 73.76 | 45.16 | 73.71 |

**Results and Discussion**

**Discussion based on D-Optimality**

The D-Optimality of equiradial designs increases for increasing axial distances for radial points n=5 and 1center point for a reduced model which is also true for full model. It was also observed that D-Optimality of equiradial design for axial distance of 1.414 shows superiority over equiradial design of 1.0 both for full and reduced model and it is true for all the radial and center points studied. The D-Optimality of equiradial design is better for reduced model than for full model for all axial distance and center points studied. This implies that equiradial design minimizes the variance of parameter estimates for reduced model than for full model. The result is comparable with what was obtained in Iwundu and Onu (2017) and Iwundu (2016b). The D-Optimality for equiradial design of 1.0 with N=6 is equal to that of N=7 for reduced model. While for n=5 with 2 center points which is equivalent to N=7 design point equals n=6 with 2 center points which is equivalent to N=8 design point, while n=5 with 4 center points (N=9 design point) equals that of n=6 with 5 center points (11 design point).

**Discussion based on G-Optimality**

The variance of prediction of equiradial designs increases with increasing center points which also increases for each radial point with 1 center point and it is irrespective of the axial distance and for 1 center point in each radial point, the variance of prediction of equiradial design of axial distance of 1.0 equals that of axial distance of 1.414. This implies that at each design point with 1 center point, the design is said to be rotatable. For full model, the maximum prediction variance of equiradial design of axial distance of 1.0 for each radial point with 1 center point is equal to that of axial distance of 1.414 and this is also true for reduced model. In full model, the maximum prediction variance equals the minimum prediction variance for each radial point with 1 center point for both axial distance of 1.0 and 1.414 same is not true for reduced model.

**Discussion based on A-Optimality**

Equiradial design for axial distance of 1.414 for both full and reduced models is found superior to equiradial design for axial distance of 1.0 on the basis of A-Optimality criterion and this is true for 1 to 5 center points and for 5 to 8 radial points considered. It was also observed that A-Optimality of equiradial designs decreases as center points increases for both axial distance of 1.0 and 1.414, this shows that A-Optimality criterion favors the addition of more center points.

**Discussion based on T-Optimality**

The T-Optimality of equiradial designs decreases with increasing center runs for full and reduced models. But for reduced model, with increasing radial points having 1 center point, it shows no steady movement, as it decreased for radial point of 6 with 1 center point and increased for radial point of 7 with 1 center point and then decreased slightly for radial point of 8 with 1 center point and this is true for axial distance of 1.0 and 1.414. The design with axial distance of 1.414 is better than the design with axial distance of 1.0 under this criterion and it is also true for both full and reduced model. For full model, equiradial designs with radial point of 8 with 1 center runs is best under T-Optimality. This criterion increases with increasing radial points and 1 center point but decreases with increasing center points.

**Discussion based on E-Optimality**

The E-Optimality criterion of equiradial designs decreases with increasing center point for both full and reduced quadratic model both for axia distance of 1.0 and 1.414. The equiradial design for 1.414 proves better than that of 1.0 under the E-Optimality criterion. The E-Optimality values for reduced model equals the E-Optimality of full model and it proves true for both axial distances of 1.0 and 1.414. it increases with radial point having 1 center point for equiradial design of 1.0 and 1.414 both for full and reduced models.

**Discussion based on D-, G-, A- and E-Efficiency criteria**

For full model, the A- and E-Efficiency for equiradial designs are 100% for both axial distancebof 1.0 and 1.414 for each radial point having 1 center point. The A- and E-Efficiency decreases as center points increases.

The D-Efficiency criterion increases slightly for increasing radial point having 1 center runs. It can be said that these values are approximately equal, that their inequality may be as a result of approximation error and it is true for both full and reduced models. Also, the D-Efficiency value for full model is approximately equal to that of reduced model for changing center points and design size. The G-Efficiency is maximum for radial point of n=5 having 1 center point but decreases for increasing center points. The value for reduced model equals the value for full model for each center points added.

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